# Space-charge waves in the wiggler field of a Raman free-electron laser

Joseph E. Willett and Bruce Bolon
*Department of Physics and Astronomy, University of Missouri–Columbia, Columbia, Missouri 65211*

Yildirim Aktas
*Department of Physics, University of North Carolina at Charlotte, Charlotte, North Carolina 28223*

Hassan Mehdian
*Department of Physics, Teacher Training University, Tehran, Iran*

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A waveguide filled with a relativistic electron beam that passes through a helical wiggler magnetic field and a uniform axial magnetic field is considered. The propagation of two types of space-charge waves in this device is analyzed. An electrostatic approximation is employed that is based on Gauss's law and the requirements that the magnetic field of the wave and the curl of the electric field of the wave both be zero in the electron-beam reference frame. These equations transformed into the laboratory reference frame are shown to be more accurate than a system of equations that includes Gauss's law in conventional form. A dispersion relation is derived with the combined effects of the wiggler and axial magnetic fields and the waveguide boundary included. Some numerical results are presented for both plasma and cyclotron types of space-charge waves. [S1063-651X(98)04406-7]

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## I. INTRODUCTION

In a free-electron laser (FEL), a relativistic electron beam radiates as a result of oscillations induced by its passage through a wiggler. The conventional wiggler is a static magnetic field that is periodic along the beam axis in the laboratory reference frame. It is a propagating electromagnetic wave in the electron-beam reference frame (beam frame). As viewed in the beam frame, the wiggler wave in a Raman FEL backscatters off of a space-charge wave. A realistic theoretical treatment of this stimulated Raman scattering process requires inclusion of the effects of the wiggler field on the propagation of the space-charge wave. Freund and Sprangle [1] have developed a theory of space-charge wave propagation through a wiggler in the presence of an axial guide magnetic field with the beam cross section assumed to be infinite. The combined effects of these two magnetic fields can be quite large as illustrated in a book by Freund and Antonsen [2].

A recent publication [3] by the authors of the present paper presents an analysis of a space-charge wave propagating through a waveguide filled with a relativistic electron beam in the presence of wiggler and axial magnetic fields. The basic equations employed in the analysis include Gauss's law in the laboratory frame. A numerical study of cyclotronlike waves with large wave numbers and small phase velocities was carried out to illustrate the combined effects of waveguide boundary and wiggler and axial magnetic fields. The purpose of the present study is to derive a dispersion relation with a wider range of validity and carry out a numerical study of both plasma and cyclotron types of space-charge waves.

The present paper contains a laboratory-frame analysis of space-charge waves that is equivalent to the beam-frame electrostatic approximation. The waves are propagating through a cylindrical metallic waveguide completely filled with a relativistic electron beam. A uniform, static axial magnetic field and a static, spatially periodic magnetic wiggler field are present. In Sec. II, equations are introduced that comprise the beam-frame Gauss's law and requirements that the magnetic field of the wave and the curl of the electric field of the wave both be zero in the beam frame. An equivalent laboratory-frame formulation that includes the linearized continuity and momentum transfer equations is applied to the analysis of space-charge waves in the absence of the wiggler to demonstrate its validity. In Sec. III, the wiggler magnetic field is represented in an idealized one-dimensional approximation and a solution of the basic laboratory-frame equations is represented as truncated Fourier and Fourier-Bessel series. A derivation of the dispersion relation is then summarized. In Sec. IV, the results of a numerical study of the effects of the waveguide boundary, wiggler field, and axial magnetic field on the two types of space-charge waves are discussed and some conclusions are presented.

## II. ELECTROSTATIC APPROXIMATION

When a space-charge wave has a phase velocity relative to the medium through which it propagates that is much smaller than the speed of light $c$, the magnetic field associated with the wave may be neglected. This is usually the case in the beam frame for a space-charge wave in a Raman FEL. Consequently, the space- and time-dependent beam-frame electric field $\delta\mathbf{E}_B$ associated with the wave satisfies the two basic differential equations of electrostatics,

$$\nabla\cdot\delta\mathbf{E}_B=4\pi\delta\rho_B \tag{1}$$

and

$$\nabla\times\delta\mathbf{E}_B=0, \tag{2}$$

 

where $\delta\rho_B$ is the charge-density perturbation associated with the wave. In this approximation (referred to herein as the electrostatic approximation), the beam-frame magnetic-field perturbation is

$$\delta\mathbf{B}_B = 0. \tag{3}$$

Three of the beam-frame Maxwell equations are satisfied, namely, Gauss's law [Eq. (1)], Faraday's law

$$\nabla\times\delta\mathbf{E}_B = -\frac{1}{c}\frac{\partial\delta\mathbf{B}_B}{\partial t}, \tag{4}$$

and Gauss's law for magnetism

$$\nabla\cdot\delta\mathbf{B}_B = 0. \tag{5}$$

The Ampère-Maxwell equation is not satisfied exactly in the electrostatic approximation and is not employed in the analysis.

It is frequently convenient to develop FEL theory in the laboratory frame. Since Gauss's law is not invariant in form under a Lorentz transformation when the Ampère-Maxwell equation is not satisfied, it must be transformed separately into the laboratory frame. This yields the modified form of Gauss's law,

$$\nabla\cdot\delta\mathbf{E} - \frac{1}{c}\mathbf{v}_\parallel\cdot\left[\nabla\times\delta\mathbf{B} - \left(\frac{4\pi}{c}\,\delta\mathbf{J} + \frac{1}{c}\frac{\partial\delta\mathbf{E}}{\partial t}\right)\right] = 4\pi\delta\rho, \tag{6}$$

where $\delta\mathbf{E}$, $\delta\mathbf{B}$, $\delta\mathbf{J}$, and $\delta\rho$ are the laboratory-frame perturbations of the electric field, magnetic field, current density, and charge density, respectively, and $\mathbf{v}_\parallel$ is the axial component of the electron beam velocity in the absence of the wave. Equation (3) may be transformed into

$$\gamma_\parallel\left(\delta\mathbf{B} - \frac{1}{c}\mathbf{v}_\parallel\times\delta\mathbf{E}\right) - \frac{\gamma_\parallel^2}{(\gamma_\parallel+1)c^2}\mathbf{v}_\parallel(\mathbf{v}_\parallel\cdot\delta\mathbf{B}) = 0, \tag{7}$$

where the Lorentz factor for the reference-frame transformation is

$$\gamma_\parallel = (1 - v_\parallel^2 c^{-2})^{-1/2}. \tag{8}$$

The two homogeneous Maxwell equations [Eqs. (4) and (5)] comprise a covariant pair that may be written in the same form in the laboratory frame, i.e.,

$$\nabla\times\delta\mathbf{E} = -\frac{1}{c}\frac{\partial\delta\mathbf{B}}{\partial t}, \tag{9}$$

$$\nabla\cdot\delta\mathbf{B} = 0. \tag{10}$$

Note the Eq. (10) is redundant for waves with a time dependence of the form $\exp(-i\omega t)$.

A laboratory-frame analysis of an axisymmetric space-charge wave in a cylindrical metallic waveguide completely filled with a relativistic electron beam will be presented. The wave will be assumed to be approximately electrostatic in the beam frame. With the beam and waveguide axis taken as the $z$ axis,

$$\mathbf{v}_\parallel = \hat{\mathbf{z}}v_\parallel. \tag{11}$$

The total electric field, magnetic field, electron density, and electron fluid velocity may be written in the form

$$\mathbf{E} = \delta\mathbf{E}, \tag{12}$$

$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}, \tag{13}$$

$$n = n_0 + \delta n, \tag{14}$$

$$\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}. \tag{15}$$

In the unperturbed state, the electric field is assumed to be negligible and the magnetic field $\mathbf{B}_0$ is the sum of a static, spatially periodic wiggler field $\mathbf{B}_\mathbf{w}$ and a uniform, static axial magnetic field $\hat{\mathbf{z}}$, $B_{0z}$; the electron density $n_0$ is uniform and constant, and the electron fluid velocity $\mathbf{v}_0$ is the sum of the transverse velocity $\mathbf{v}_\mathbf{W}$ (due to passage through the wiggler) and the uniform, constant axial velocity $\hat{\mathbf{z}}v_\parallel$. The charge-density perturbation $\delta\rho$ and the linearized current density perturbation $\delta\mathbf{J}$ are given by

$$\delta\rho = -e\,\delta n \tag{16}$$

and

$$\delta\mathbf{J} = -e(n_0\delta\mathbf{v} + \mathbf{v}_0\delta n), \tag{17}$$

where $-e$ is the electron charge. Equations (6) and (7) may be expressed in the form

$$\nabla\cdot\delta\mathbf{E} - \frac{v_\parallel}{c}\left[\hat{\mathbf{z}}\cdot\nabla\times\delta\mathbf{B} + \left(\frac{4\pi e}{c}\right)(n_0\delta v_z + v_\parallel\delta n) - \frac{1}{c}\frac{\partial\delta E_z}{\partial t}\right]$$
$$= -4\pi e\,\delta n, \tag{18}$$

$$\delta\mathbf{B} - \frac{v_\parallel}{c}\hat{\mathbf{z}}\times\delta\mathbf{E} - \frac{\gamma_\parallel v_\parallel^2}{(\gamma_\parallel+1)c^2}\,\delta B_z\hat{\mathbf{z}} = \mathbf{0}. \tag{19}$$

The basic equations also include Eq. (9) and the linearized continuity and momentum transfer equations

$$\frac{\partial\delta n}{\partial t} + n_0\nabla\cdot\delta\mathbf{v} + (\nabla\delta n)\cdot\mathbf{v}_0 = \mathbf{0}, \tag{20}$$

$$\frac{\partial\delta\mathbf{v}}{\partial t} + \mathbf{v}_0\cdot\nabla\delta\mathbf{v} + \delta\mathbf{v}\cdot\nabla\mathbf{v}_0 = -e(\gamma_0 m_0)^{-1}[\delta\mathbf{E} - c^{-2}\mathbf{v}_0\mathbf{v}_0\cdot\delta\mathbf{E}$$
$$+ c^{-1}\delta\mathbf{v}\times\mathbf{B}_0 + c^{-1}\mathbf{v}_0\times\delta\mathbf{B}$$
$$- \gamma_0^2 c^{-3}(\mathbf{v}_0\times\mathbf{B}_0)\mathbf{v}_0\cdot\delta\mathbf{v}], \tag{21}$$

where

$$\gamma_0 = (1 - v_0^2 c^{-2})^{-1/2}. \tag{22}$$

Although the wiggler induced velocity and axial velocity are, in general, relativistic, the space-charge oscillation velocity $\delta\mathbf{v}$ is assumed to be nonrelativistic.

To illustrate the significance of using the modified form of Gauss's law rather than the conventional form, space-charge wave propagation through the beam-filled waveguide will first be analyzed with no wiggler present. In this case, the unperturbed magnetic field, electron fluid velocity, and Lorentz factor reduce to

$$\mathbf{B}_0 = \hat{\mathbf{z}} B_0, \tag{23}$$

$$\mathbf{v}_0 = \hat{\mathbf{z}} v_\parallel, \tag{24}$$

$$\gamma_0 = \gamma_\parallel. \tag{25}$$

A solution of the foregoing linear equations can be expressed in the form

$$\delta\mathbf{E} = \hat{\mathbf{r}}\delta E_r + \hat{\mathbf{z}}\delta E_z, \tag{26}$$

$$\delta E_r = \delta\hat{E}_r J_1(p_{0v} r/R)\exp[i(kz - \omega t)], \tag{27}$$

$$\delta E_z = \delta\hat{E}_z J_0(p_{0v} r/R)\exp[i(kz - \omega t)], \tag{28}$$

$$\delta\mathbf{B} = \hat{\theta}\delta B_\theta, \tag{29}$$

$$\delta B_0 = \delta\hat{B}_\theta J_1(p_{0v} r/R)\exp[i(kz - \omega t)], \tag{30}$$

$$\delta n = \delta\hat{n} J_0(p_{0v} r/R)\exp[i(kz - \omega t)], \tag{31}$$

$$\delta\mathbf{v} = \hat{\mathbf{r}}\delta v_r + \hat{\theta}\delta v_\theta + \hat{\mathbf{z}}\delta v_z, \tag{32}$$

$$\delta v_r = \delta\hat{v}_r J_1(p_{0v} r/R)\exp[i(kz - \omega t)], \tag{33}$$

$$\delta v_\theta = \delta\hat{v}_\theta J_1(p_{0v} r/R)\exp[i(kz - \omega t)], \tag{34}$$

$$\delta v_z = \delta\hat{v}_z J_0(p_{0v} r/R)\exp[i(kz - \omega t)]. \tag{35}$$

Here $J_0$ and $J_1$ are Bessel functions of the first kind of order 0 and 1, respectively, $p_{0\nu}$ (with $\nu = 1,2,3,...$) is the $\nu$th zero of $J_0$, and $R$ is the electron beam radius and waveguide inner radius. With the unperturbed beam velocity $v_\parallel$, wave number $k$, and wave angular frequency $\omega$ taken to be positive, the beam velocity and phase velocity are in the positive $z$ direction in the laboratory frame. Substitution of Eqs. (26)–(35) into Eqs. (9), (18), (19), (20), and (21) leads to seven linear, homogeneous algebraic equations in the seven unknown amplitudes $\delta\hat{E}_r$, $\delta\hat{E}_z$, $\delta\hat{B}_\theta$, $\delta\hat{n}$, $\delta\hat{v}_r$, $\delta\hat{v}_\theta$, and $\delta v_z$. The necessary and sufficient condition for a nontrivial solution yields the laboratory-frame dispersion relation

$$\frac{p_{0v}^2}{\gamma_\parallel^2 (k - \omega v_\parallel c^{-2})^2 R^2} + \frac{(\Omega_0^2 - \bar{\omega}^2)(\bar{\omega}^2 - \omega_p^2 \gamma_\parallel^{-2})}{\bar{\omega}^2(\Omega_0^2 + \omega_p^2 \gamma_\parallel^{-2} - \bar{\omega}^2)} = 0, \tag{36}$$

where $\Omega_0$ and $\omega_p$ are the laboratory-frame cyclotron frequency and plasma frequency given by

$$\Omega_0 = eB_{0z}/(\gamma_0 m_0 c), \tag{37}$$

$$\omega_p = \left(\frac{4\pi e^2 n_0}{\gamma_0 m_0 c}\right)^{1/2} \tag{38}$$

with $\gamma_0$ set equal to $\gamma_\parallel$ since the wiggler is absent, and

$$\bar{\omega} = \omega - k v_\parallel. \tag{39}$$

The above dispersion relation can be transformed into the beam frame by use of

$$\omega_B = \gamma_\parallel(\omega - k v_\parallel), \tag{40}$$

$$k_B = \gamma_\parallel(k - \omega v_\parallel c^{-2}), \tag{41}$$

$$\omega_{pB} = \omega_p, \tag{42}$$

$$\Omega_{0B} = \gamma_\parallel \Omega_0, \tag{43}$$

where $\omega_B$, $k_B$, $\omega_{pB}$, and $\Omega_{0B}$ are the wave angular frequency, wave number, plasma frequency, and cyclotron frequency in the beam frame. The result is

$$\frac{p_{0v}^2}{k_B^2 R^2} + \frac{(\Omega_{0B}^2 - \omega_B^2)(\omega_B^2 - \omega_{pB}^2)}{\omega_B^2(\Omega_{0B}^2 + \omega_{pB}^2 - \omega_B^2)} = 0, \tag{44}$$

which is the well-known beam-frame dispersion relation. If the conventional Gauss's law were employed rather than the modified form, the procedure would yield

$$\frac{p_{0v}^2}{k(k - \omega v_\parallel c^{-2}) R^2} + \frac{(\Omega_0^2 - \bar{\omega}^2)(\bar{\omega}^2 - \omega_p^2 \gamma_\parallel^{-2})}{\bar{\omega}^2(\Omega_0^2 + \omega_p^2 \gamma_\parallel^{-2} - \bar{\omega}^2)} = 0 \tag{45}$$

in the laboratory frame, which transforms to

$$\frac{p_{0v}^2}{k_B^2 R^2} + \frac{(1 + \omega_B k_B^{-1} v_\parallel c^{-2})(\Omega_{0B}^2 - \omega_B^2)(\omega_B^2 - \omega_{pB}^2)}{\omega_B^2(\Omega_{0B}^2 + \omega_{pB}^2 - \omega_B^2)} = 0 \tag{46}$$

in the beam frame. This contains the erroneous factor $(1 + \omega_B k_B^{-1} v_\parallel c^{-2})$.

## III. DISPERSION RELATION WITH WIGGLER PRESENT

An analysis of an axisymmetric space-charge wave in a wiggler magnetic field will be presented next. As in Sec. II, a uniform axial magnetic field is also present and the relativistic electron beam completely fills the cylindrical metallic waveguide. The beam-frame electrostatic approximation is invoked. Although the analysis is carried out in the laboratory frame, the basic equations are equivalent to the beam-frame Gauss's law and the requirements that the magnetic field of the wave and the curl of the electric field of the wave both be zero in the beam frame. The linearized continuity and momentum transfer equations are also employed.

In the unperturbed state, the electron density $n_0$ is uniform and constant, the electric field $\mathbf{E}_0$ is assumed to be negligible, and the magnetic field $\mathbf{B}_0$ and electron velocity $\mathbf{v}_0$ are given by

$$\mathbf{B}_0 = \hat{\mathbf{r}} B_W \cos\Theta + \hat{\theta} B_W \sin\Theta + \hat{\mathbf{z}} B_{0z}, \tag{47}$$

$$\mathbf{v}_0 = \hat{\mathbf{r}} v_W \cos\Theta + \hat{\theta} v_W \sin\Theta + \hat{\mathbf{z}} v_\parallel. \tag{48}$$

Here $B_W$ is the magnitude of the wiggler magnetic field, $B_{0z}$ is the axial magnetic field, $\Theta$ is defined as

$$\Theta = k_W z - \theta, \tag{49}$$

$v_W$ is the transverse electron velocity given by

$$v_W = \Omega_W v_\parallel / (\Omega_0 - k_W v_\parallel), \tag{50}$$

$v_\parallel$ is the axial electron velocity, $\Omega_W$ is the relativistic cyclotron frequency corresponding to the wiggler field given by

$$\Omega_W = eB_W / (\gamma_0 m_0 c), \tag{51}$$

and $\gamma_0$ is the Lorentz factor given by

$$\gamma_0 = [1 - (v_W^2 + v_\parallel^2) c^{-2}]^{-1/2}. \tag{52}$$

The quantities $B_W$, $B_{0z}$, $v_W$, $v_\parallel$, $k_W$, $\Omega_W$, $\Omega_0$, and $\gamma_0$ are independent of position and time.

The small-amplitude wave causes a perturbation that is assumed to be of the form

$$\delta E_r = \delta\hat{E}_{r0} J_1(p_{0v} r/R) \exp[i(kz - \omega t)], \tag{53}$$

$$\delta E_z = \delta\hat{E}_{z0} J_0(p_{0v} r/R) \exp[i(kz - \omega t)], \tag{54}$$

$$\delta B_\theta = \delta\hat{B}_{\theta 0} J_1(p_{0v} r/R) \exp[i(kz - \omega t)], \tag{55}$$

$$\delta n = \delta\hat{n}_0 J_0(p_{0v} r/R) \exp[i(kz - \omega t)], \tag{56}$$

$$\delta v_r = (\delta\hat{v}_{r0} + \delta\hat{v}_{r1} \cos\Theta + \delta\hat{v}_{r2} \sin\Theta) J_1(p_{0v} r/R) \times \exp[i(kz - \omega t)], \tag{57}$$

$$\delta v_\theta = (\delta\hat{v}_{\theta 0} + \delta\hat{v}_{\theta 1} \cos\Theta + \delta\hat{v}_{\theta 2} \sin\Theta) J_1(p_{0v} r/R) \times \exp[i(kz - \omega t)], \tag{58}$$

$$\delta v_z = \delta\hat{v}_{z0} J_0(p_{0v} r/R) \exp[i(kz - \omega t)]. \tag{59}$$

These perturbation quantities are functions of $\Theta$ that are represented by Fourier series with only the dominant terms retained. The quantities $\delta E_z$, $\delta n$, and $\delta v_z$, which vanish at $r = R$, are functions of $r$ that are represented by Fourier-Bessel series with only the dominant terms retained. The radial dependences of $\delta E_r$, $\delta B_\theta$, $\delta v_r$, and $\delta v_\theta$ are then represented in a consistent manner. In these truncated series, the dominant terms are assumed to be those that survive in the limit of infinite waveguide radius or in the limit of zero wiggler field.

Substitution of Eqs. (53)–(59) into Eqs. (9), (18), (19), (20), and (21) results in the following eleven linear homogeneous algebraic equations in the eleven unknown amplitudes:

$$p_{0v} R^{-1} \gamma_\parallel^2 (\delta\hat{E}_{r0} - v_\parallel c^{-1} \delta\hat{B}_{\theta 0}) + i\gamma_\parallel^2 (k - \omega v_\parallel c^{-2}) \delta\hat{E}_{z0} + 4\pi e \delta\hat{n}_0 - 4\pi e \gamma_\parallel^2 n_0 v_\parallel c^{-2} \delta\hat{v}_{z0} = 0, \tag{60}$$

$$\delta\hat{B}_{\theta 0} - v_\parallel c^{-1} \delta\hat{E}_{r0} = 0, \tag{61}$$

$$ik \delta\hat{E}_{r0} + p_{ov} R^{-1} \delta\hat{E}_{z0} - i\omega c^{-1} \delta\hat{B}_{\theta 0} = 0, \tag{62}$$

$$-i\overline{\omega} \delta\hat{n}_0 + n_0 p_{0v} R^{-1} \delta\hat{v}_{r0} + ikn_0 \delta\hat{v}_{z0} = 0, \tag{63}$$

$$-i\overline{\omega} \alpha_1^{-1} \delta\hat{v}_{z0} + (\Omega_\omega/2)(\delta\hat{v}_{r2} - \delta\hat{v}_{\theta 1}) + \gamma_0^{-1} \gamma_\parallel^{-2} \alpha_1^{-1} (e/m_0) \delta\hat{E}_{z0} = 0, \tag{64}$$

$$-i\overline{\omega} \delta\hat{v}_{r0} + [\Omega_0 + (1/2)\eta] \delta\hat{v}_{\theta 0} + (1/2) p_{0v} \alpha_1^{-1} R^{-1} v_W \delta\hat{v}_{r1} - (1/2) \alpha_2 \alpha_1^{-1} R^{-1} v_W \delta\hat{v}_{\theta z} + (e/m_0) \gamma_0^{-1} \times [1 - (1/2) v_W^2 c^{-2}] \delta\hat{E}_{r0} - (e/m_0) \gamma_0^{-1} c^{-1} v_\parallel \delta\hat{B}_{\theta 0} = 0, \tag{65}$$

$$-i\overline{\omega} \delta\hat{v}_{\theta 0} - [\Omega_0 + (1/2)\eta] \delta\hat{v}_{r0} + (1/2) p_{0v} \alpha_1^{-1} R^{-1} v_W \delta\hat{v}_{\theta 1} + (1/2) \alpha_2 \alpha_1^{-1} R^{-1} v_W \delta\hat{v}_{r2} = 0, \tag{66}$$

$$-i\overline{\omega} \delta\hat{v}_{r1} + v_W \alpha_1^{-1} (p_{0v} - \alpha_2) R^{-1} \delta\hat{v}_{r0} + [k_W v_\parallel + (1/4)\eta] \delta\hat{v}_{r2} + [\Omega_0 + (1/4)\eta] \delta\hat{v}_{\theta 1} - v_W v_\parallel \alpha_1^{-1} \gamma_0^{-1} c^{-2} (e/m_0) \delta\hat{E}_{z0} = 0, \tag{67}$$

$$-i\overline{\omega} \delta\hat{v}_{r2} - v_W \alpha_2 \alpha_1^{-1} R^{-1} \delta\hat{v}_{\theta 0} - [k_W v_\parallel - (1/4)\eta] \delta\hat{v}_{r1} + \alpha_1^{-1} (\eta v_\parallel v_W^{-1} - k_W v_W - \Omega_W) \delta\hat{v}_{z0} + [\Omega_0 + (3/4)\eta] \delta\hat{v}_{\theta z} = 0, \tag{68}$$

$$-i\overline{\omega} \delta\hat{v}_{\theta 1} + v_W (p_{0v} - \alpha_2) \alpha_1^{-1} R^{-1} \delta\hat{v}_{\theta 0} + [k_W v_\parallel - (1/4)\eta] \delta\hat{v}_{\theta z} + \alpha_1^{-1} (\Omega_W + k_W v_W - \eta v_\parallel v_W^{-1}) \delta\hat{v}_{z0} - [\Omega_0 + (3/4)\eta] \delta\hat{v}_{r1} = 0, \tag{69}$$

$$-i\overline{\omega} \delta\hat{v}_{\theta 2} + v_W \alpha_2 \alpha_1^{-1} R^{-1} \delta\hat{v}_{r0} - [k_W v_\parallel + (1/4)\eta] \delta\hat{v}_{\theta 1} - [\Omega_0 + (1/4)\eta] \delta\hat{v}_{r2} - v_W v_\parallel \gamma_0^{-1} c^{-2} (e/m_0) \alpha_1^{-1} \delta\hat{E}_{z0} = 0. \tag{70}$$

Here

$$\eta = -k_W v_\parallel \gamma_0^2 v_W^2 c^{-2}, \tag{71}$$

$$\alpha_1 = 2R^{-2} [J_1(p_{0v})]^{-2} \int_0^R r J_0(p_{0v} r/R) J_1(p_{0v} r/R) dr, \tag{72}$$

$$\alpha_2 = 2R^{-1} [J_1(p_{0v})]^{-2} \int_0^R J_0(p_{0v} r/R) J_1(p_{0v} r/R) dr. \tag{73}$$

The necessary and sufficient condition for a nontrivial solution of Eqs. (60)–(70) may, after some extensive algebraic manipulation, be cast into the form

$$\frac{p_{0v}^2}{\gamma_\parallel^2 (k - \omega v_\parallel c^{-2})^2 \rho^2 R^2} + \frac{(b^2 \Omega_0^2 - \overline{\omega}^2)(\overline{\omega}^2 - \omega_b^2 \Phi \gamma_0^{-1} \gamma_\parallel^{-2})}{\overline{\omega}^2 (b^2 \Omega_0^2 + \omega_b^2 \Psi \gamma_0^{-1} \gamma_\parallel^{-2} - \overline{\omega}^2)} = 0, \tag{74}$$

where

$$\omega_b = (4\pi e^2 n_0 / m_0)^{1/2}, \tag{75}$$

$$\rho = \{1 + (\Omega_W \Omega_0 v_W v_\parallel^{-1} - \delta_1)[\Omega_0 - k_W v_\parallel{}^2 - \overline{\omega}^2]^{-1}\}^{1/2}, \tag{76}$$

$$b = 1 - k_W v_\parallel v_W^2 c^{-2} \gamma_0^2 \Omega_0^{-1}, \tag{77}$$

$$\Psi = 1 - (1/2)\gamma_{\parallel}^2 v_W^2 c^{-2}, \tag{78}$$

and

$$\Phi = 1 - \gamma_{\parallel}^2 \Omega_0 \Omega_W v_W v_{\parallel}^{-1} [(v_{\parallel} v_W^{-1} \Omega_W + v_W^2 v_{\parallel}^{-2} \Omega_0) v_{\parallel} v_W^{-1} \Omega_W - (\overline{\omega}^2 + \delta_1)]^{-1} - \delta_2 [(\Omega_0 - k_W v_{\parallel})^2 + \Omega_W \Omega_0 v_W v_{\parallel}^{-1} - (\overline{\omega}^2 + \delta_1)]^{-1}. \tag{79}$$

The quantities $\delta_1$ and $\delta_2$ vanish in the limit of infinite waveguide radius and also in the limit of zero wiggler field. They are given by a hierarchy of algebraic equations that will be omitted for brevity. Equation (74) is the laboratory-frame dispersion relation for space-charge waves in the wiggler. The range of validity of this equation exceeds that of the corresponding dispersion relation derived in Ref. [3] since the present derivation is not based on the approximations

$$1 - \omega k^{-1} v_{\parallel} c^{-2} \cong \gamma_{\parallel}^{-2} \tag{80}$$

and

$$1 + \omega_B k_B^{-1} c^{-1} \cong 1. \tag{81}$$

Note that $\omega_b$ is the nonrelativistic beam plasma frequency in the laboratory frame.

## IV. NUMERICAL RESULTS AND DISCUSSION

The laboratory-frame analysis presented herein is appropriate for two types of space-charge waves in a beam-filled waveguide containing wiggler and axial magnetic fields. The beam is treated as neutralized in the unperturbed state and the waves are treated as electrostatic in the beam frame. Consequently the present analysis, transformed to the beam frame with the wiggler field set equal to zero, yields the same results as the quasistatic analysis of space-charge waves in a plasma-filled waveguide by Trivelpiece and Gould [4]. Equation (44) is a quadratic equation for the square of the frequency $\omega_B^2$ as a function of the square of the wave number $k_B^2$ for axisymmetric space-charge waves in the absence of a wiggler field. Choosing the minus sign in the quadratic formula for $\omega_B^2$ yields the dispersion relation for plasma waves. The frequency of each mode increases monotonically with increasing wave number $k_B$ from zero at $k_B = 0$ and approaches the plasma frequency $\omega_{pB}$ or the cyclotron frequency $\Omega_{0B}$, whichever is lower, as $k_B$ approaches infinity. The phase velocity $\omega_B/k_B$ is generally sufficiently small compared to the speed of light $c$, so that the electrostatic approximation is valid for plasma space-charge waves. Choosing the plus sign in the quadratic formula for $\omega_B^2$ yields the dispersion relation for cyclotron waves. The frequency $\omega_B$ of each mode decreases monotonically with increasing $k_B$ from the upper hydrid frequency $(\Omega_{0B}^2 + \omega_{pB}^2)^{1/2}$ at $k_B = 0$ and approaches $\Omega_{0B}$ or $\omega_{pB}$, whichever is higher, as $k_B$ approaches infinity. Thus, in the electrostatic approximation, the cyclotron space-charge modes are backward waves, i.e., they have oppositely directed phase and group velocities. The electrostatic assumption restricts the validity of the theory to low phase velocities and frequencies below the cutoff of the empty waveguide. The electrostatic approximation is valid at large wave numbers for which $\omega_B/k_B \ll c$ and also for weak magnetic fields $(\Omega_{0B} \ll \omega_{pB})$, where the cyclotron wave resembles a Langmuir wave for which the electron current tends to cancel the effects of the displacement current. Otherwise, a fully electromagnetic treatment is required as given, e.g., by Ivanov and Alexov [5].

At large wave numbers, it is convenient to employ an alternative classification of the two types of space-charge waves. The waves that, in the infinite-wave-number limit and in the absence of the wiggler field, have frequencies approaching the cyclotron frequency $\Omega_{0B}$ will be referred to as cyclotronlike; those that have frequencies approaching the plasma frequency $\omega_{pB}$ will be referred to as plasmalike. In the limit of infinite normalized beam-frame wave number $(k_B R \to \infty)$, the present theory yields

$$\omega_B^2 = \gamma_{\parallel}^2 b^2 \Omega_0^2 \tag{82}$$

for the cyclotronlike waves and

$$\omega_B^2 = \omega_b^2 \Phi_{\infty} \gamma_0^{-1}, \tag{83}$$

where, with the limit $R \to \infty$ also imposed,

$$\Phi_{\infty} = 1 - \gamma_{\parallel}^2 \Omega_0 \Omega_W v_W v_{\parallel}^{-1} \times [(v_{\parallel} v_W^{-1} \Omega_W + v_W^2 v_{\parallel}^{-2} \Omega_0) v_{\parallel} v_W^{-1} \Omega_W - \omega_b^2 \Phi_{\infty} \gamma_0^{-1} \gamma_{\parallel}^{-2}]^{-1}, \tag{84}$$

for the plasmalike waves. Note that $b$ times $B_{0z}$ is the effective axial magnetic field and $\Phi_{\infty}$ times $n_0$ is the effective electron density in the presence of the wiggler field.

Numerical calculations have been made to illustrate the effects of waveguide radius, wiggler magnetic field, and axial magnetic field on both types of space-charge waves with large beam-frame wave numbers. Wiggler magnetic field $B_W$ and wiggler wavelength $2\pi/k_W$ were taken to be 760 G and 5 cm, respectively. The inner radius $R$ of the beam-filled waveguide was taken to be 0.3 cm. Laboratory frame electron density $n_0$ was taken to be $10^{12}$ $\mathrm{cm}^{-3}$ and electron-beam energy $(\gamma_0 - 1) m_0 c^2$ was taken to be 700 keV corresponding to a Lorentz factor $\gamma_0$ of 2.37. Axial magnetic field $B_{0z}$ was varied from 0 to 25.4 kG, which corresponds to a variation from 0 to 5 in the normalized laboratory-frame relativistic cyclotron frequency $\Omega_0/(ck_\omega)$ associated with $B_{0z}$. The first $(\nu = 1)$ mode was chosen for which $p_{0\nu} = p_{01} = 2.405$.

Figures 1–4 show the normalized beam-frame frequency $\omega_B/(ck_W)$ of the cyclotronlike and plasmalike waves as functions of $\Omega_0/(ck_W)$, which hereafter will be referred to as the normalized axial magnetic field. Both group-I stable orbits $(\Omega_0 < k_W v_{\parallel})$ and group-II orbits $(\Omega_0 > k_W v_{\parallel})$ are considered. The normalized beam-frame wave number was taken as $k_B R = \infty$ (circles), $k_B R = 100$ (solid curve), and $k_B R = 10$ (dashed curve). In the invariant phase $k_B z_B - \omega_B t_B = kz - \omega t$, $\omega_B$ may be positive or negative with $k_B$, $k$, and $\omega$ taken as positive. The calculations were made for a wave propagating in the negative $z_B$ direction and, consequently, $\omega_B$ is negative. The minus sign is omitted in the figures.

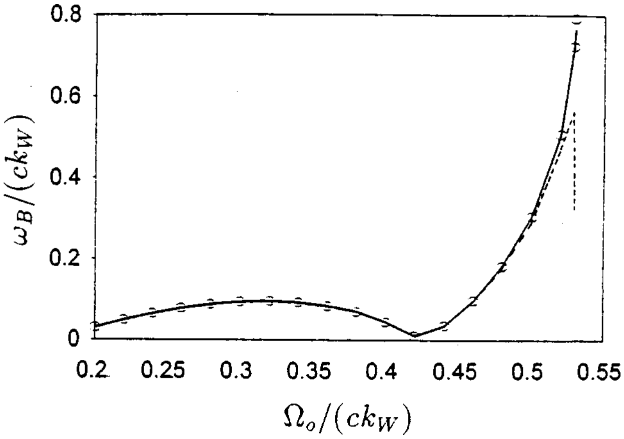


FIG. 1. Normalized beam-frame frequency $\omega_B/(ck_W)$ of the cyclotronlike wave as a function of the normalized axial magnetic field $\Omega_0/(ck_W)$ for group-I orbits. The values of the normalized beam-frame wave number $k_BR$ are $\infty$ (circles), 100 (solid curve), and 10 (dashed curve).

Figures 1 and 2 illustrate the variation of the frequency of the cyclotronlike wave with axial magnetic field $B_{0z}$ for group-I and group-II orbits, respectively. The circles correspond to $k_BR=\infty$ and were computed using Eq. (82). The frequency, which would be proportional to $B_{0z}$ in the absence of the wiggler, is modified by wiggler effects manifested through the factors $b$ and $\gamma_\parallel$. The frequencies computed for $k_BR=100$ using the complete dispersion relation [Eq. (79)] are in close agreement with those for $k_BR=\infty$ for most values of $\Omega_0/(ck_\omega)$. An exception occurs in Fig. 1 when $\Omega_0/(ck_\omega)$ approaches 0.53 (where group-I orbits become unstable) due to a reduction in the radius factor $\rho$ and the effective normalized beam-frame wave number $k_B\rho R$. Exceptions also occur in Fig. 2 at some values of $\Omega_0/(ck_W)\leqslant 1$, where the wave is not cyclotronlike for $k_BR$ finite. Larger departures from the circles were found for $k_BR=10$ as expected.

Figures 3 and 4 illustrate the variation of the frequency of the plasmalike waves with axial magnetic field $B_{0z}$ for group-I and group-II orbits, respectively. The circles correspond to $k_BR=\infty$ and were computed using Eq. (83) with $\Phi_\infty$ computed using Eq. (84) in the infinite-waveguide-radius approximation, for which $\delta_1=\delta_2=0$. The rate of change of the electron axial velocity with electron energy is proportional to a function $\Phi_0$ that is equal to density factor $\Phi$ [defined by Eq. (79)] with $\delta_1$, $\delta_2$, and $\overline{\omega}(=\omega_B\gamma_\parallel^{-1})$ set to equal zero. For group-I orbits, $\Phi_0\cong 1$ for $B_{0z}$ small and rises abruptly, approaching infinity as $B_{0z}$ is increased to the value that results in orbital instability $[\Omega_0/(ck_w)=0.53]$. It is important to note that, unlike $\Phi_0$, the density factor $\Phi$ does not become singular as the maximum value of $B_{0z}$ for group-I orbit stability is approached. Consequently, in Fig. 3 the frequency of the plasmalike wave does not become large as $\Omega_0/(ck_\omega)$ approaches 0.53. In Fig. 4, no frequencies are shown for $\Omega_0/(ck_\omega)<1.2$; the plasmalike wave is unstable in this negative-mass regime (where $\Phi_0<0$) not only for $k_BR=\infty$ but for $k_BR=100$ and $k_BR=10$ as well. The plasmalike waves with $k_BR=100$ (solid curve) and $k_BR=\infty$

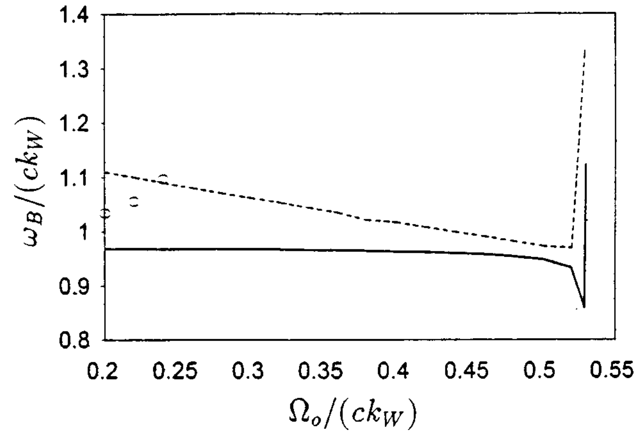


FIG. 3. Normalized beam-frame frequency $\omega_B/(ck_W)$ of the plasmalike wave as a function of the normalized axial magnetic field $\Omega_0/(ck_W)$ for group-I orbits. The values of the normalized beam-frame wave number $k_BR$ are $\infty$ (circles), 100 (solid curve), and 10 (dashed curve). The waveguide radius $R$ is also infinite when $k_BR$ is infinite.

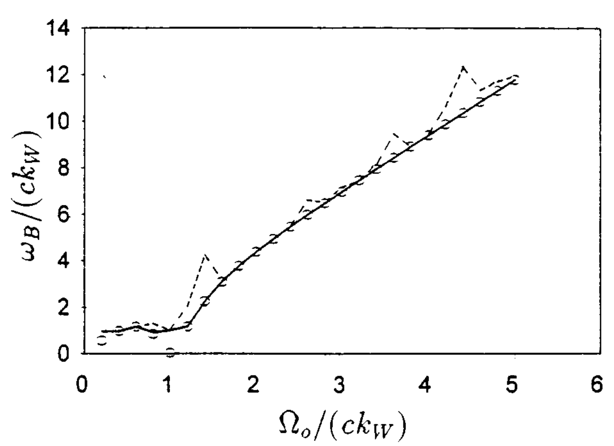


FIG. 2. Normalized beam-frame frequency $\omega_B/(ck_W)$ of the cyclotronlike wave as a function of the normalized axial magnetic field $\Omega_0/(ck_W)$ for group-II orbits. The values of the normalized beam-frame wave number $k_BR$ are $\infty$ (circles), 100 (solid curve), and 10 (dashed curve).

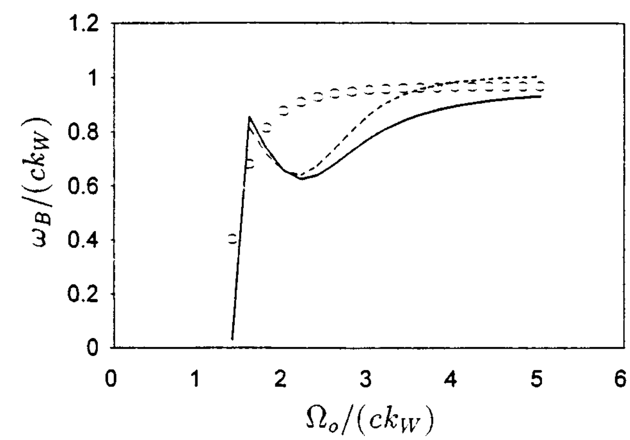


FIG. 4. Normalized beam-frame frequency $\omega_B/(ck_W)$ of the plasmalike wave as a function of the normalized axial magnetic field $\Omega_0/(ck_W)$ for group-II orbits. The values of the normalized beam-frame wave number $k_BR$ are $\infty$ (circles), 100 (solid curve), and 10 (dashed curve). The waveguide radius $R$ is also infinite when $k_BR$ is infinite.

(circles) have significantly different frequencies since $R = 3$ mm for the former case and $R=\infty$ for the latter case. Frequencies computed for $k_B R = 100$ using Eq. (74) with $\rho$ and $\Phi$ given by Eqs. (76) and (79) in the infinite-waveguide-radius approximation, for which $\delta_1 = \delta_2 = 0$, however, agree with those computed using Eqs. (83) and (84) to three significant figures. It is interesting to note that Eq. (84) is a quadratic equation for $\Phi_\infty$. The smaller root was used in Figs. 3 and 4. The larger root predicts stable waves with $k_B R = \infty$ and $R = \infty$ for $\Omega_0/(ck_\omega) < 0.26$ and for $\Omega_0/(ck_\omega) < 2.2$ with group-I and group-II orbits, respectively.

A system of laboratory-frame equations for the electric field, magnetic field, electron density, and electron velocity was introduced herein that is equivalent to the system of beam-frame equations in the electrostatic approximation. The validity of this new system was demonstrated by deriving the dispersion relation for space-charge waves in a waveguide filled with a relativistic electron beam. It was shown that an error would have resulted if the conventional form of Gauss's law had been employed in the laboratory frame. The new system of equations was then applied to obtain the dispersion relation for the space-charge waves in the presence of a magnetic wiggler field. The resulting dispersion relation was cast into the form it would assume in the absence of the wiggler field but with the electron density, axial magnetic field, and waveguide radius replaced by effective values modified by the wiggler. In general, calculation of these effective values requires the solution of a long chain of algebraic equations, which will be presented elsewhere [6]. Numerical calculations were made for space-charge waves with finite beam-frame wave numbers and finite beam and waveguide radius. The negative mass instability was found, but no singularity of the electron density factor $\Phi$ was found.

Free-electron laser theories may be developed in either the electron-beam reference frame or the laboratory reference frame. Equivalent results may be obtained in the two frames provided that the two sets of basic equations are entirely equivalent. It was shown herein that electrostatic analyses based on the conventional form of Gauss's law in both frames are not equivalent. Errors in the dispersion relation for space-charge waves are only of second order in $\omega_B/k_B c$ when derived in the beam frame, but are of first order when derived in the laboratory frame for a strongly relativistic beam. The present analysis was carried out in the laboratory frame using basic equations that were entirely equivalent to the basic equations of the beam frame. Indentical results could have been derived in the beam frame using the conventional form of Gauss's law, but this would have required treating the wiggler field as a propagating electromagnetic wave.

The present theory can be used to compute the dispersion relation for a space-charge wave in an FEL wiggler. Numerical results thereby obtained could be compared with Raman FEL experiments. The theory developed herein is based on the assumption that the electron beam completely fills the waveguide. Since this cannot be achieved experimentally, an experiment could be performed with the ratio of the beam radius to the waveguide inner radius as near unity as is feasible. Measurements of the radiation frequency, electron-beam energy, electron density, wiggler wavelength, waveguide inner radius, and axial magnetic field would be required. Calculations of the radiation frequency could then be made using the phase-matching conditions and dispersion relations for the space-charge wave and the electromagnetic wave in the wiggler. This would determine if the radial waveguide boundary conditions improve the agreement with experimental results.

In order to use the results of previous Raman FEL experiments, the present authors plan to extend the theory to the case of a partially filled waveguide. Typical values of the ratio of the beam radius to the waveguide inner radius range from 0.2 to 0.35 (see, e.g., Refs. [7–10]). Studies of the effects of the wiggler on the electromagnetic wave are also underway. The purpose of this research is to provide a method of obtaining theoretical values for observable quantities such as the radiation frequency and the growth rate based on realistic treatment of the space-charge and electromagnetic waves in the Raman FEL wiggler.


[1] H. P. Freund and P. Sprangle, Phys. Rev. A **28**, 1835 (1983).

[2] H. P. Freund and J. M. Antonsen, Jr., *Principles of Free-Electron Lasers* (Chapman and Hall, London, 1992), Chaps. 2 and 4.

[3] J. E. Willett, Y. Aktas, and B. Bolon, Phys. Plasmas **3**, 2441 (1996).

[4] A. W. Trivelpiece and R. W. Gould, J. Appl. Phys. **30**, 1784 (1959).

[5] S. T. Ivanov and E. G. Alexov, J. Plasma Phys. **43**, 51 (1990).

[6] B. Bolon, Ph.D. thesis, University of Missouri–Columbia, 1999.

[7] J. Fajans and G. Bekefi, Phys. Fluids **29**, 3461 (1986).

[8] Li-Yi Lin, T. C. Marshall, and M. A. Cecere, Nucl. Instrum. Methods Phys. Res. A **341**, 265 (1994).

[9] A. A. Kaminsky, A. K. Kaminsky, V. P. Sarantsev, S. N. Sedykh, A. P. Sergeev, and A. A. Silivra, Nucl. Instrum. Methods Phys. Res. A **341**, 105 (1994).

[10] M. C. Wang, Z. Wang, J. Chen, Z. Lu, and L. Zhang, Nucl. Instrum. Methods Phys. Res. A **304**, 116 (1990).